Analysis of Seepage with Nonlinear Permeability Using Finite Difference Method and Comparison with Least Square Finite Element Method

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Abstract

In most geotechnical analyses, soil properties are assumed to be spatially and temporally invariant; however, these soil parameters usually vary from point to point and also may vary in time. Therefore in this research, the coefficients of permeability are assumed to vary in terms of geometry, external load influences and the effect of head variation in the system and the resulted nonlinear seepage problem is solved using Finite Difference Method. The seepage Problem is analyzed for two cases of variable and constant coefficients of permeability. The results are also compared with the Least Square FE Method.

Keywords: Dams, Seepage, Consolidation, Nonlinearity, Anisotropy.

Introduction

Free-surface seepage problems have been attracting interests of many engineers and mathematicians due to the strong non-linearity as well as the importance in designing the hydraulic structures, such as embankments, canals, and earth and rock-fill dams. Determination of free-surface profile, velocity, and pressure distributions is of prime importance in a free surface seepage analysis. Free-surface seepage flow is governed by an elliptic partial differential equation when steady state flow conditions have been considered. Solution of this elliptic partial differential equation may be carried out analytically and numerically. Analytical solutions of this equation require several assumptions such as ideal solution domains or homogeneous material properties. On the other hand, numerical solutions have to be used if the solution domain has complicated geometry and/or inhomogeneous material properties. Among the numerical solution techniques finite difference and finite element method are perhaps the most popular. In most geotechnical analyses, soil properties are assumed to be spatially and temporally invariant and thus, average property values are used. In reality, however, these soil parameters usually vary from point to point (heterogeneous) and even at one point they may have different values in various measured directions (anisotropy). Moreover, these parameters may vary in time while a geotechnical process is in progress due to an external influence such as surface pressure or due to the change of chemical compositions. Computational process related to flow problems using numerical techniques, usually homogeneous conditions are assumed for the coefficient of permeabilities and anisotropic conditions are assumed throughout. In this research, the coefficients of permeability are assumed to vary in term of geometry, external load influences such as those causing consolidation effects, and the effect of head variation in the system where seepage is taking place. In order to define these variations, two conditions are introduced in this paper. The first condition can be explained by, for instance, an embankment load over a confined saturated fine grain soil layer. This load would begin to consolidate underlying materials. At the end of consolidation process, the permeability of the materials is changed and can be described by a governing differential equation, which can lead to a solution. The second condition to be discussed is the variations of the head can also have an effect on the consolidation process resulting in permeability variations. This effect can be seen in Terzaghi's effective stress equation. The influence of head variation is introduced by a specific function, which can be solved numerically. This changes the governing differential equation to a nonlinear one. A numerical solution is required in such cases [1]. In the present study, the finite difference formulation is used in order to solve the

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nonlinear governing differential equation. The objective of the finite difference method for solving a partial differential equation (PDE) is generally to transform a mathematic problem into an algebraic problem.
Finite difference analysis has been implemented in a number of areas in engineering such as heat transfer [3], structural analysis [4], hydrodynamics [5,6], wave problems [7,8] as well as geotechnical interests such as seepage [9-13], consolidation [14] and liquefaction [15]. It also works well with anisotropic materials [16].
So that, in this research, the finite difference method is used and the non-linear seepage is solved by using this method and then the results are compared with least square finite element method done by Toufigh [1].

Variability of the coefficient of permeability

In flow problems, both the magnitude and direction of governing fluid flows are highly sensitive to the coefficient of the permeability. For simplicity, this parameter is usually assumed to be a constant in space and time. In this study, the coefficient of permeability is assumed to be spatially variable. The variation of coefficient of permeability was defined for different cases, and then the resulted governing differential equation was solved. In order to define a function for the variation of the permeability two conditions were introduced.
The first condition is a simple condition where the coefficient of the permeability is a function of material properties and geometrical conditions. From most classical soil mechanics literature studied, it is well known that coefficient of permeability is directly proportional to the void ratio of the soil. As the void ratio increases or decreases, so does the coefficient of permeability, Lambe and Whitman [17]. Only confined flow was considered here. These coefficients of permeability would also be time dependent as long as the consolidation process continues. In order to define good estimates for coefficient of permeability one should undertake laboratory testing to define the equations for k_x and k_y, the coefficients of permeability in x and y direction, respectively. The variation of k_x can be simply expressed by any order binominal equation, which in this study was considered to be second order.

$$k_x = a_x x^2 + b_x x + c_x \tag{1}$$

Where a_x, b_x and c_x, are the coefficients that can be determined from a curve fitting procedure based on the results from laboratory and field-testing. Similarly k_y can be expressed by a similar second order binominal equation. Generally k_y can vary by either the effect of overburden pressure of the natural soils or the influence of excess stresses due to an embankment load. For the first case, as the overburden pressure increases with depth, there would be a tendency for the material to become more compacted, therefore reducing k_y with depth. For the second case, as the depth increases the effect of embankment load decreases i.e. less consolidation, and thus k_y increases. The effect of the second imposed condition is opposite to the first case, and these physical effects with depth should be superimposed in order to define Equation (1) for every starting point at interface of embankment and natural soil in the vertical direction. The second condition to be introduced here is that the coefficient of permeability, in addition to the first case, can be affected by the variation of heads in the upstream, downstream, or in the soil. In the next section the related relationships are described.

Relationship Between Effective Stress and Soil Void Ratio

In the soil consolidation process, the relationship between effective stress and void ratio can be demonstrated in e vs. log p space, Figure 1, Leroueil et al. [18]. The first portion of the curve with lower slope, which is due to unloading of the sample, is not considered here. Only the second portion with slope of c_c, which is mainly due to loading, is considered. The void ratio "e" of the material at any stage of the consolidation can be determined by:

$$e = c_c \log \frac{\sigma'}{\sigma_1'} + e_1 \tag{2}$$

where σ' is the applied effective stresses (head) corresponding to e and σ_1' is the known effective stress corresponding to e_1. Equation (3) can be written as:

$$e = a \log \sigma' + b \tag{3}$$

where $a = c_c$ and $b = -c_c \log \sigma_1' + e_1$

Relationship Between Void Ratio and Coefficient of Permeability

It can be observed from previous research of Lambe and Whitman [17], Leroueil [18] and Cedergren [19] that the relationship between void ratio and logarithm of coefficient of permeability is linear, Figure 1. Similar to previous case, "e" void ratio of the material at any stage can be determined by:

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$$e = c_k \log \frac{k}{k_1} + e_1 \tag{4}$$

where c_k is the slope of the curve, k is the unknown coefficient of permeability corresponding to e, and k_1 is the known coefficient of permeability corresponding to e_1. By rearranging the Equation (4), the coefficient of permeability can be found as follows:

$$\log k = \frac{e}{c_k} - \frac{e_1}{c_k} + \log k_1 \tag{5}$$

Since $\frac{-e_1}{c_k} + \log k_1$ is a constant value assumed to be equal to d, and $c = \frac{1}{c_k}$, therefore:

$$\log k = c\, e + d \tag{6}$$

And finally k can be written as:

$$k = 10^{(c\, e + d)} \tag{7}$$

And with substitution of equation (3) into equation (7), it can be written as:

$$k = 10^{(\alpha \log \sigma' + \beta)} \tag{8}$$

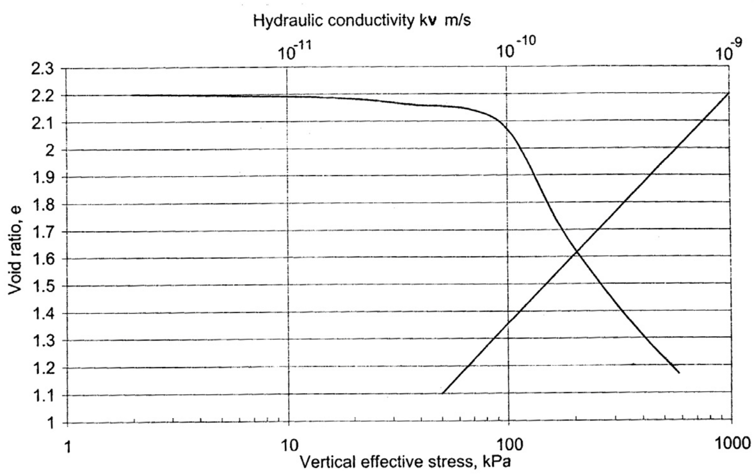

Figure 1- Typical e against $\log \sigma'$ and e against $\log k$ curve (after Leroueil et al.)

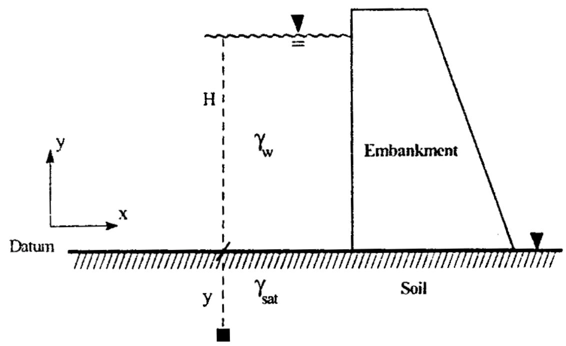

Figure 2- Schematic diagram of an embankment dam.

where α is equal to $c\,a$ and β is equal to $c\,b + d$,which all of the parameters a. b, c, and d are constant and can be determined from laboratory or in-situ testing.

Relationship Between Heads and Coefficient of Permeability

From the effective stress Terzaghi's Equation and from the information in Figure 2, the effective stress at any point can be written as:

$$\sigma' = (-y\, \gamma_{sat} + \gamma_w H) - (h + y)\gamma_w \tag{9}$$

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where γ_{sat} is the saturated density of the soil, h is the total head, $h + y$ is the pressure head, H is the upstream water height and γ_w is water density.
By substituting Equation (9) into Equation (8), it can be written as:

$$k = 10^{\alpha \log[-y\,\gamma_{sat} + (H-h-y)\gamma_w] + \beta} \tag{10}$$

Equation (10) can be simplified to

$$k = 10^{\beta}\,[-y\,\gamma_{sat} + (H-h-y)\gamma_w]^{\alpha} \tag{11}$$

In the above equation α, β, γ_{sat} and γ_w are constants that depend on material properties and can be determined from laboratory or in-situ testing. The value of total head h depends on the geometry of the considered point and is an unknown value, H is the height of water at upstream and y is the depth of the considered point from datum. It can be concluded from the above equation that at any point within the confined flow the coefficient of permeability can be defined as a function of total head h which will directly influence the solution of the governing differential equation [1].

Non-linear Governing Seepage Equation and Related Finite Difference Formulation

The 2-D governing equation of water flow in porous media, where Darcy's law is applicable is given by:

$$\frac{\partial}{\partial x}(\gamma_w k_x \frac{\partial h}{\partial x}) + \frac{\partial}{\partial y}(\gamma_w k_y \frac{\partial h}{\partial y}) = 0 \tag{12}$$

The above equation can be simplified by assuming γ_w, water density, to stay constant at all times, and therefore:

$$\frac{\partial}{\partial x}(k_x \frac{\partial h}{\partial x}) + \frac{\partial}{\partial y}(k_y \frac{\partial h}{\partial y}) = 0 \tag{13}$$

Under conditions of homogeneity, k_x and k_y are assumed to be constants which do not vary in space. In addition, applying anisotropy conditions requires $k_x \neq k_y$. Generally for simplicity k_x and k_y are assumed to be constant and for more simplicity, they are assumed to be equal and constant. However, in this research these coefficients are assumed to be variable which would change the differential equation to a non-linear one. k_x and k_y can now be expressed by Equation (1) and Equation (11) and can be substituted in Equation (13).

In seepage problems, in addition to evaluation of heads at various locations in the system, three other parameters are important to be evaluated. These are total discharge rate, exit hydraulic gradient, and uplift pressure. These parameters are known as the secondary solutions. The exit hydraulic gradient can be evaluated by:

$$i_x = \frac{\Delta h}{\Delta x} \tag{14}$$

Total discharge rate can be calculated on the bases of discharge for each element at any section, and the summation of these discharge rates would be the total discharge rate of the system, which will be:

$$Q = -\sum_{i=1}^{N} d_i\, k_{xi}\, i_{xi} \tag{15}$$

where d, is the distance between two adjacent nodes, with the value i_{xi} as the hydraulic gradient for that nodes. The exit hydraulic gradient would be known at the downstream section of the system. Uplift pressure can be calculated on the bases of Bernoulli's equation by knowing total head (h) from analysis and evaluation of the concerning point from geometry assuming, that v2/2g =0 [1].

Fundamental Formulation

Equation (13) can be written in difference form of:

$$\frac{1}{\Delta x^2}(H_{i+1,j}\,k_x + H_{i-1,j}\,k_x - 2H_{i,j}\,k_x) + \frac{1}{\Delta y^2}(H_{i,j+1}\,k_y + H_{i,j-1}\,k_y - 2H_{i,j}\,k_y) = 0 \tag{16}$$

If $\Delta x = \Delta y$ then equation (16) can be simplified as:

$$H_{i+1,j}\,k_x + H_{i-1,j}\,k_x + H_{i,j+1}\,k_y + H_{i,j-1}\,k_y - 2H_{i,j}\,(k_x + k_y) = 0 \tag{17}$$

And if $k_x = k_y$ then the above relation would be more simplified as:

$$H_{i+1,j} + H_{i-1,j} + H_{i,j+1} + H_{i,j-1} - 4H_{i,j} = 0 \tag{18}$$

For vertical boundaries, where $\partial h/\partial x = 0$, we have $H_{i+1,j} = H_{i-1,j}$ and for horizontal boundaries, where

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$\partial h/\partial y = 0$, we have $H_{i,\,j+1} = H_{i,\,j-1}$. For nodes in datum level in upstream, $H_{i,j} = H_0$ and for nodes in datum level in downstream, $H_{i,\,j} = 0$

Numerical Examples

In this section two examples are provided for the proposed types of variations on k_x and k_y.

Example 1: To illustrate the proposed methods, consider Example 18.2 of Lambe and Whitman [17]. A schematic diagram of a concrete dam is given in Figure 3. This system consists of two sheet piles of 21 meters height at upstream and downstream of the dam. In order to analyses the problem, the permeable section of the system was divided into 67 nodes. Sheet piles were considered as impermeable boundaries, where $\partial h / \partial x = 0$ and other impermeable boundaries where $\partial h / \partial y = 0$ are at the bottom of the 64 meter thick permeable layer and dam itself. A thirty-meter distance away from the system (sheet piles) was chosen as a limit for numerical analysis where it assumed there is no flow-taking place away from these limits in the permeable layer. Variable heads at upstream and downstream locations were chosen in order to examine the effect of the proposed solution.

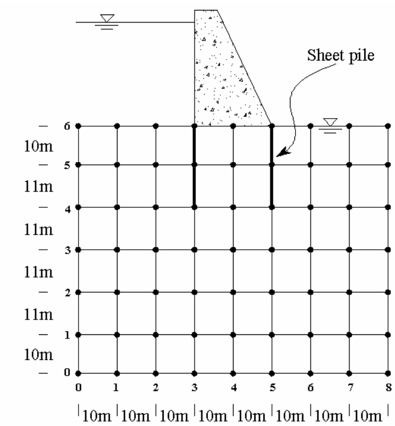

Figure 3- Schematic diagram of the system and FD mesh.

In order to apply Equation (11), the following values were used based on Effati [20].

$\alpha = -0.034049,\ \beta = -1.0,\ \gamma_{sat} = 22\ kN/m^3,\ \gamma_w = 10\ kN/m^3$

Results for head, the coefficient of permeability *k* and discharge rate were obtained based on the above values employing finite difference formulation. Comparing the results of the heads obtained here with flow nets in Lambe and Whitman [17] shows only very small differences. Any conclusions based on head results and flow nets alone may not be justified due to the accuracy of the results. A flow net drawing is based on a trial and error procedure and is not affected by upstream or downstream heads. In Figure 4 variation of coefficient of permeability *k* is shown against head (water height) in the upstream. It is clear from this figure that, (i) when the head at any node varies, it would influence the permeability of that node, (ii) as the head increases the values of permeability decrease, and (iii) when the head at any point increases, consolidation of the material occurs resulting in reduced the permeability. The variation of *k* against head is non-linear because the proposed function for *k* in Equation (11) is non-linear. This figure also shows the comparison between FD method and Least Square FE method. As it is seen, both methods show decrease of k with increase of head and the differences are in an acceptable range.

Figure 5 shows the variation of the discharge rate under the dam against upstream head. Two types of curves are shown in Figure 5, one with constant permeability and the other with variable permeability (proposed method). In the one with constant permeability, similar to most classical seepage problems, permeability is assumed constant throughout the analysis and the system, and if it varies, it is not due to the effect of upstream head. In this case *k* was assumed to be 0.080 cm/sec. But in the other one variable *k* refers to the influence of head on discharge rate. It can be seen from Figure 5 that for both cases when upstream head increases, the discharge rate also increases. It should be clarified, however, that in the actual case, head effects influenced permeability. The discharge rate is different from that of the constant permeability case. The difference would be higher for longer values of upstream head, i.e. for h = 120 m the effect of the head on discharge rate is about 2%. This is mainly due to the effect of upstream head on permeability. Also, finite difference method and least square finite element method are compared with each other. It is observed that FD method gives lower discharge than LSFE method and both methods show the same pattern.

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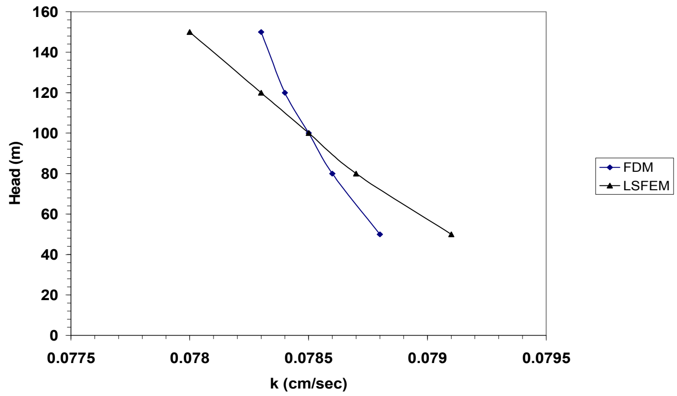

Figure 4- Comparison between FDM and LSFEM for variable *k*.

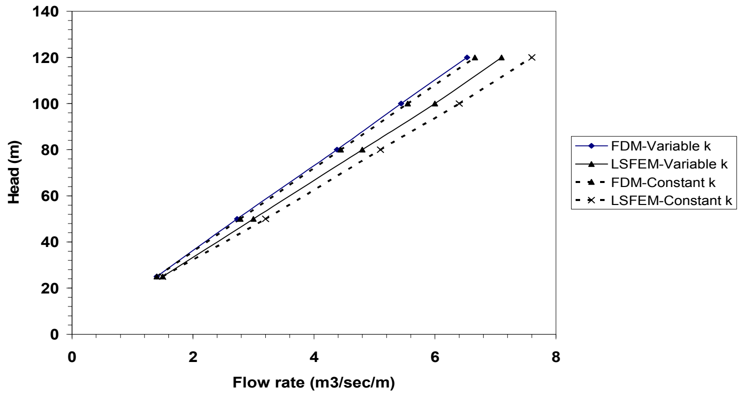

Figure 5- Comparison of discharge rate variation under dam against head for constant and variable coefficient of permeability.

Example 2: In these example variations of k_x and k_y are not effected by a direct influence of head but they are based on other effects using Equation (1). A schematic diagram of a concrete dam is given in Figure 6.

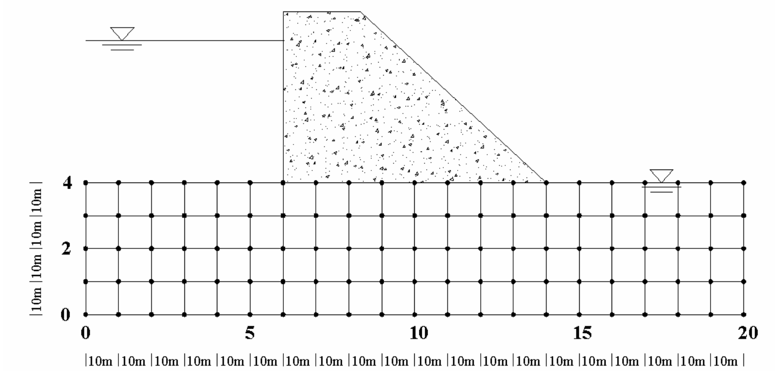

Figure 6- Schematic diagram of the system with FD mesh.

The permeable section of the system was divided into 105 nodes. The top and bottom portion of the permeable section with thickness of 40 meters were considered as impermeable boundaries, where $\partial h / \partial y = 0$. Sixty meters from the toe and heel of the dam were chosen as a limit for numerical analysis where no flow was assumed to take place away from these limits in the permeable layer. The proposed variations for k_x and k_y, based on Equation (1) and Effati [20], are as follows:

$$k_x = 0.375 \times 10^{-3}\, x^2 - 0.375 \times 10^{-2}\, x + 10; \qquad at\ x = 0.0m \qquad k_x = 10\ m/day$$

$$k_y = 0.255 \times 10^{-2}\, y^2 - 0.375 \times 10^{-2}\, y + 2.5; \qquad at\ y = 0.0m \qquad k_y = 2.5\ m/day$$

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In this example results for exit gradient and uplift pressure are presented based on above values for k_x and k_y using finite difference formulation. Figure 7 shows the exit gradient in vertical direction against upstream head for constant and variable permeability based on data in this example.

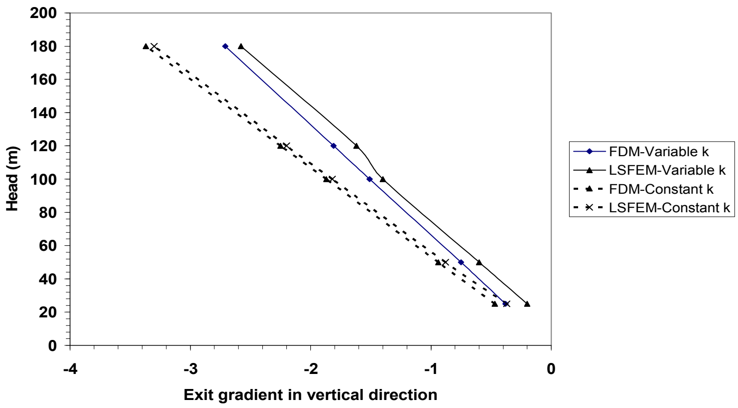

Figure 7- Variation of exit hydraulic gradient against head for constant and variable permeability.

For low upstream head the difference between constant and variable permeability conditions is sometimes negligible, but as the upstream head increases, in large dams the difference becomes more significant which might influence the design of the whole system. For the upstream height of 180 meters, the exit gradient difference is about 24%, which would reduce the factor of safety against piping to a low point, which, in turn, may result in changing the geometry of the dam. It should be noted that constant values of permeability considered in the computation would result in higher values for exit gradient, which would be on the safer side. This figure also shows the comparison of FD method and LSFE method for variable *k*. As it is seen, both methods show increase of exit gradient (absolute value) with head and the results are in good agreement.
Figure 8 shows the uplift pressure at the bottom of the dam against upstream head for constant and variable permeability based on data in this example. As the head in the upstream of the dam will increase, obviously the uplift pressure under dam increases, but as it can be seen, there is a difference between constant or variable permeability conditions. This difference would be around 8% for an upstream head of 150 meters. Again, it should be noted that with constant permeability the value of uplift pressure is higher than that with the variable permeability, which would also be on the safe side.

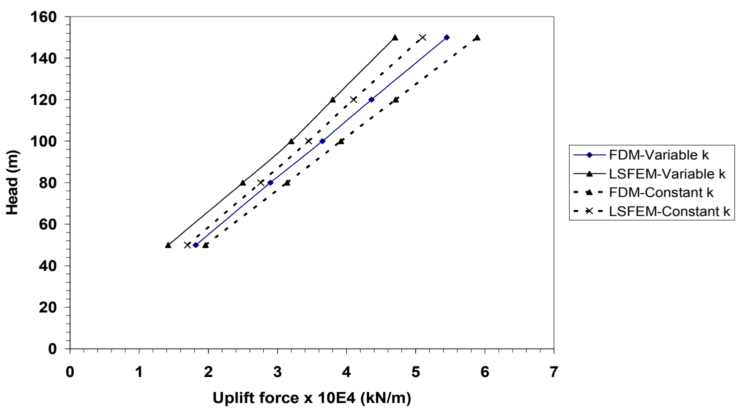

Figure 8- Comparison of the uplift pressure against head for constant and variable *k*.

Figure 8 also shows the uplift pressure against upstream head for FD method and LSFE method. It is observed that FD method gives higher uplift pressure than the LSFE method.

Conclusions

This paper presents a non-linear governing differential equation for a confined seepage problem under non-homogeneous and anisotropic conditions. This non-linear performance is introduced by the governing equation

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based on actual material behavior and solving the resulting non-linear differential equation numerically using the finite difference formulation. The results are also compared with least square finite element method and they show a very good agreement with each other.
Comparison of the results for discharge rate between constant and variable permeability conditions shows little effect of low head on discharge rate results. However, as upstream head increases, the effect of variable permeabilities becomes more significant. Usually the difference in discharge rate between variable and constant permeability for a typical head is not more than 8%. Results of exit gradient for a critical condition show that the values are less affected for low head, but this effect increases for higher head. The results of assuming variable permeability conditions, would give a lower safety factor regarding piping, etc. In terms of uplift pressure, as the head increases, the uplift pressure also increases. But there is only a slight difference between uplift pressures under constant or variable permeability conditions for any given head. In general, the effect of variable coefficient of permeability may not be significant on small dams, but as the height of the dam increases, the effect becomes more considerable. It is believed that this would influence the geometry and design of the dam and therefore variable permeability analysis should be conducted. Finite difference method and least square finite element method results were in good agreement and the differences between the results are negligible. Therefore, finite difference method, because of its more simplicity and less resource consuming, is a preferable method for dealing with (non-linear) seepage problems.

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