Buckling Analysis of Composite Laminates Using the Element Free Galerkin Method

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Abstract
An element free Galerkin method (EFG) is used to analyze composite laminates, considering Kirchoff plate theory. Moving least square approximation (MLS) is used to construct shape functions using a set of nodes scattered in the problem domain. Numerical examples are presented for computing buckling loads of composite laminates using EFG method with different conditions.
Keywords: Element free Galerkin method, moving least square approximation, composite laminates,

Introduction

Nowadays, composite laminates are used in many structures because of their low weight, high strength, and high stiffness. In designing composite structures computing buckling force of laminates is very important. therefore ,numerical methods such as finite element method(FEM)[1], have been used for analyzing composite laminates. Mesh-free methods, are new numerical methods for analyzing structures. the smooth particle hydrodynamics (SPH)[2], the reproducing kernel particle method (RKPM)[3], the element free Galerkin method (EFG)[4,5,6,7,8], the meshless local Petrov-Galerkin method (MLPG)[9], and the h-p clouds method[10], are some mesh-free methods that have been used to analyze structures. EFG is an effective mesh-free method which uses moving least square approximation (MLS) to construct shape functions. In this paper we have used EFG method to solve the equations of static buckling of laminates. As will be seen, the deflection in the z direction is the only unknown at a node. So, the dimension of the discrete equations obtained by the EFG method is only one third of that in FEM method. To examine the efficiency of the EFG method, buckling factor of composite laminates has been calculated.

Static buckling equations of composite laminates

A composite laminate with thickness h in the z direction is shown in fig.1. the laminated plate is consist of n_l layers. The reference plane, z=0, is located at the undeformed neutral plane of the laminated plate. The direction of fibers in a layer is indicated by α fig.2.The laminated plate is subjected to in-plane forces within the plane of symmetry of the plate on its edges. So, we have:

$$N_x=-N_0 \tag{1}$$
$$N_y=-\mu_1 N_0 \tag{2}$$
$$N_{xy}=-\mu_2 N_0 \tag{3}$$

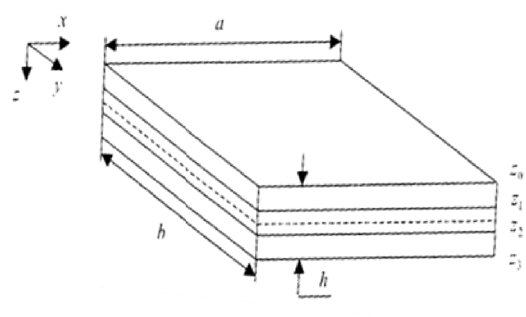

Figure 1-coordinate system of a composite laminate

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Which, μ_1 and μ_2 are functions of coordinates, and N_0 is a constants. The strain energy of bending of the laminates is written by follow [11],[12],

$$\Pi_b=1/2\int_A \varepsilon_p{}^t\sigma_p\, dA \tag{4}$$

Where A stands for the area of the plate, ε_p is the pseudo-strain, and σ_p is the pseudo-stress. For analyzing thin composite laminates, we have used classic plate theory (CPT), known as kirchoff plate theory. CPT assumes that normal to the neutral surface of undeformed plate remain straight and normal to the neutral surface during deformation. This assumption is called the kirchoff assumption, that results in :

$$\varepsilon_{xz}=0 \tag{5}$$
$$\varepsilon_{yz}=0 \tag{6}$$

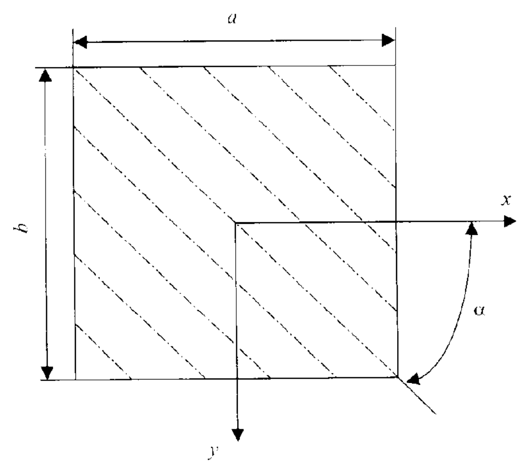

Figure 2- fiber orientation of α in a laminate

so, we should only compute ε_x , ε_y, and ε_{xy}. Therefore we can use equations of plain stress. Displacements in the X and Y direction, u and v, at a distance z from the neutral surface can be expressed by

$$u=-z\frac{\partial w}{\partial x} \tag{7}$$

$$v=-z\frac{\partial w}{\partial y} \tag{8}$$

where w is the deflection of the middle plane of the plate in the z direction. Using above equations we can write:

$$u=\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}=\left\{-z\frac{\partial}{\partial x} \;\; -z\frac{\partial}{\partial y} \;\; 1\right\}^T w=L_u w \tag{9}$$

the relationship between the three components of strain and the deflection can be given by:

$$\varepsilon_x=\frac{\partial u}{\partial x} =-z\frac{\partial^2 w}{\partial x^2} \tag{10}$$

$$\varepsilon_y=\frac{\partial v}{\partial y} =-z\frac{\partial^2 w}{\partial y^2} \tag{11}$$

$$\varepsilon_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=-2z\frac{\partial^2 w}{\partial x\partial y} \tag{12}$$

or in matrix form:

$$\varepsilon=zLw \tag{13}$$

where, ε is the vector of in-plane strains, and L is the differential operator matrix given by:

$$L=\begin{Bmatrix} -\frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial y^2} \\ -\frac{\partial^2}{\partial x \partial y} \end{Bmatrix} \quad (14)$$

Pseudo-strain and pseudo-stress, are defined by[13]:

$$\varepsilon_p = zLw \quad (15)$$

$$\sigma_p = c\varepsilon = zcLw \quad (16)$$

$$c = E/(1-\nu^2)\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & \vdots 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \quad (17)$$

substituting above equations into equation(4),results in

$$\pi_b = 1/2\int_A \left[D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_{66}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + 4\left(D_{16}\frac{\partial^2 w}{\partial x^2} + D_{26}\frac{\partial^2 w}{\partial y^2}\right)\frac{\partial^2 w}{\partial x \partial y}\right] dA \quad (18)$$

the strain energy caused by in-plane forces is expressed by

$$\pi_i = 1/2\int_A \left[N_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + N_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] dA \quad (19)$$

elastic constants of laminates (D_{ij}), are defined as follows

$$D_{ij} = 1/3\sum_{k=1}^{n_l}(\bar{Q}_{ij})_k(z_k^3 - z_{k-1}^3), \quad i,j=1,2,6 \quad (20)$$

$$\bar{Q}_{11} = Q_{11}\cos^4\alpha + 2(Q_{12} + 2Q_{66})\sin^2\alpha\cos^2\alpha + Q_{22}\sin^4\alpha \quad (21)$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\alpha\cos^2\alpha + Q_{12}(\sin^4\alpha + \cos^4\alpha) \quad (22)$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})\sin\alpha\cos^3\alpha + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\alpha\cos\alpha \quad (23)$$

$$\bar{Q}_{22} = Q_{11}\sin^4\alpha + 2(Q_{12} + 2Q_{66})\sin^2\alpha\cos^2\alpha + Q_{22}\cos^4\alpha \quad (24)$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})\sin^3\alpha\cos\alpha + (Q_{12} - Q_{22} + 2Q_{66})\sin\alpha\cos^3\alpha \quad (25)$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\alpha\cos^2\alpha + Q_{66}(\sin^4\alpha + \cos^4\alpha) \quad (26)$$

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} \quad (27)$$

$$Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} \quad (28)$$

$$Q_{22}=\frac{E_2}{1-\nu_{12}\nu_{21}} \tag{29}$$

$$Q_{66}=G_{12} \tag{30}$$

$$\nu_{21}E_1=\nu_{12}E_2 \tag{31}$$

where ν_{12} and ν_{21} are Poisson's ratios parallel to and perpendicular to the orientation of fibers, E_1 and E_2 are corresponding Young's moduli, and α is the angle of fiber orientation of the ply. the total potential energy of the composite laminate, both strain energies of bending and in-plane forces, is:

$$\Pi=\Pi_b+\Pi_i \tag{32}$$

MLS shape functions for composite laminates

As we mentioned before, only the deflection w is independent. Displacements in the X and Y direction, u and v, can be obtained from w. so, only the deflection w needs to be approximated. MLS approximation has been used as shape function for deflection approximation. A set of nodes are required to construct deflection approximation. The deflection approximation function $w^h(x)$ has a polynomial form of[14],

$$w^h(x) = p^T(x)a(x). \tag{33}$$

we use quadratic polynomial basis for p:

$$p^T(x)=\left\{1, x, x^2, xy, y^2\right\} \tag{34}$$

MLS shape functions are used to determine coefficients a(x). a weighted discrete norm, J, is defined:

$$J=\sum_{I=1}^{n} w(X-X_I)[p^T(x_I)a(x)-w_I]^2 \tag{35}$$

Where $w(X-X_I)$ is the weight function of point x. minimization of J results in the following equation:

$$A(x)a(x) = B(x)w \tag{36}$$

The matrices A and B are denoted as follows:

$$A(x) = \sum_{I=1}^{n} w(X-X_I)P(X_I)P^T(X_I) \tag{37}$$

$$B(X)=\{w(X-X_1)P(X_1),\ldots,w(X-X_n)P(X_n) \tag{38}$$

Substituting above equations into equation (33) yields

$$w^h(x) = \sum_{I=1}^{n}(P^T(X)A^{-1}(X)B(X))_I w_I= \sum_{I=1}^{n} \Phi_I(X)w_I \tag{39}$$

where, w_I is the nodal displacement in the z direction, and Φ_I is the shape functions, which are obtained as,

$$\Phi_I(x) = \sum_{j}^{m} p_j(X)(A(X)^{-1}B(X))_{Ji} = P^TA^{-1}B_I \tag{40}$$

In order to establish static buckling equations of laminates, the first and second order partial derivatives of the shape function $\Phi_I(x)$ should be computed. We define γ(X) to compute partial derivatives efficiently,

$$\gamma(X) = A^{-1}(X)P(X). \tag{41}$$

So, we can write:

$$\Phi(X) = \gamma^T(X)B(X), \quad (42)$$

$$A\boldsymbol{\gamma} = P \quad (43)$$

The partial derivatives can be computed as follows

$$A\boldsymbol{\gamma}_{,x} = P_{,x}\text{-}A_{,x}\gamma \quad (44)$$

$$A\boldsymbol{\gamma}_{,y} = P_{,y}\text{-}A_{,y}\gamma \quad (45)$$

$$A\gamma_{,xx} = P_{,xx} - (A_{,xx}\gamma+2A_{,x}\gamma_{,x}) \quad (46)$$

$$A\gamma_{,xy} = P_{,xy} - (A_{,xy}\gamma+A_{,x}\gamma_{,y}+ A_{,y}\gamma_{,x}) \quad (47)$$

$$A\gamma_{,yy} = P_{,yy} - (A_{,yy}\gamma+2A_{,y}\gamma_{,y}) \quad (48)$$

$$\Phi_{I,x} = \gamma^T_{,x}B_I+ \gamma^T B_{I,x} \quad (49)$$

$$\Phi_{I,y} = \gamma^T_{,y}B_I+ \gamma^T B_{I,y} \quad (50)$$

$$\Phi_{I,xx} = \gamma^T_{,xx}B_I+ 2\gamma^T_{,x}B_{I,x}+ \gamma^T B_{I,xx} \quad (51)$$

$$\Phi_{I,xy} = \gamma^T_{,xy}B_I + \gamma^T_{,x}B_{I,y}+ \gamma^T_{,y}B_{I,x}+ \gamma^T B_{I,xy} \quad (52)$$

$$\Phi_{I,yy} = \gamma^T_{,yy}B_I+ 2\gamma^T_{,y}B_{I,y}+ \gamma^T B_{I,yy} \quad (53)$$

The support domain of a point is defined using a weight function. For classical thin composite laminates, the weight function should be non-negative and its first and second derivatives must be non-singular. So we use a quartic spline weight function as follows:

$$w(r)= \begin{cases} \left(1 - 6r^2 + 8r^3 - 3r^4\right) & for\ 0 \le r \le 1, \\ 0 & for\ r > 1 \end{cases} \quad (54)$$

Or

$$w(r)= \begin{cases} \left(1 - \frac{47}{10}r^2 + 12r^4 - 10r^5 + \frac{1}{2}r^6 + \frac{6}{5}r^7\right) & r \le 1, \\ 0 & r > 1 \end{cases} \quad (54)$$

Where, r, is the normalized distance:

$$r=\frac{\left\| x-x_I \right\|}{d_s} \quad (55)$$

$$d_s= \alpha_s d_c \quad (56)$$

where, α_s is a constant, and d_c is the maximum distance between point X and nodes X_I. in this paper we take α_s=3.9.

Enforcing boundary conditions

MLS shape functions do not satisfy the Kronecker delta condition, $\Phi_I(x_J) = \delta_{IJ}$, so, the boundary conditions have to be imposed. In this paper we have used, Lagrange multipliers to enforc boundary conditions. The discretized boundary conditions with Lagrange multipliers are given below[5];

$$\int_{\Gamma u} \delta\lambda^T (w- \tilde{w})d\Gamma = 0 \tag{57}$$

w= $\tilde{w}$ on essential boundary
where λ is a vector of Lagrange multipliers which is interpolated as follows:

$$\lambda(x)=N_I(s)\lambda_I, \quad x\in \Gamma_u \tag{58}$$

where s and $N_I(s)$ are, the arc length and Lagrange interpolant along the boundary. The variation of the Lagrange multipliers can be written as[5],

$$\delta\lambda(x) = N_I(s)\lambda_I , \ x\in \Gamma_u \tag{59}$$

$$N_I(s)= \frac{(s-s_0)(s-s_1)...(s-s_n)}{(s_I-s_0)(s_I-s_1)...(s_I-s_n)} \tag{60}$$

the deflection at any point in the composite laminate can be approximated using equation 39. Total energy can be obtained by substituting equation 39 into equations 18 and 19. The stationary condition for the potential energy gives:

$$\frac{\partial n}{\partial w_i}=0 \tag{61}$$

Which, leads to a set of discrete eigenvalue equations for the composite laminates:

$$\left[K - N_0 B\right] W = 0 \tag{62}$$

Where

$$W = \left\{w_1, w_2, ..., w_{nt}\right\}^T \tag{63}$$

Is the deflection vector and K is the global stiffness matrix for the composite laminate, which is assembled using nodal stiffness.

$$K_{ij}=\frac{1}{2}\int_A \Big[2D_{11}\frac{\partial^2\Phi_i}{\partial x^2}\frac{\partial^2\Phi j}{\partial x^2} + 2D_{12}\Big(\frac{\partial^2\Phi_i}{\partial x^2}\frac{\partial^2\Phi j}{\partial y^2} + \frac{\partial^2\Phi j}{\partial x^2}\frac{\partial^2\Phi i}{\partial y^2}\Big)+2D_{22}\frac{\partial^2\Phi_i}{\partial y^2}\frac{\partial^2\Phi j}{\partial y^2}+8D_{66}\frac{\partial^2\Phi_i}{\partial x\partial y}\frac{\partial^2\Phi j}{\partial x\partial y}+$$
$$4D_{16}\Big(\frac{\partial^2\Phi i}{\partial x^2}\frac{\partial^2\Phi j}{\partial x\partial y}+\frac{\partial^2\Phi j}{\partial x^2}\frac{\partial^2\Phi_i}{\partial x\partial y}\Big) + 4D_{26}\Big(\frac{\partial^2\Phi_i}{\partial y^2}\frac{\partial^2\Phi j}{\partial x\partial y}+\frac{\partial^2\Phi j}{\partial y^2}\frac{\partial^2\Phi_i}{\partial x\partial y}\Big)\Big]dA \tag{64}$$

$$B_{ij} = \frac{1}{2}\int_A \left[2\frac{\partial \Phi_i}{\partial x}\frac{\partial \Phi j}{\partial x} + 2\mu_1\frac{\partial \Phi_i}{\partial y}\frac{\partial \Phi j}{\partial y} + 2\mu_2\left(\frac{\partial \Phi_i}{\partial x}\frac{\partial \Phi j}{\partial y} + \frac{\partial \Phi j}{\partial x}\frac{\partial \Phi_i}{\partial y}\right)\right] dA \quad (65)$$

Eigenvalue equations of static buckling of composite laminates

By using orthogonal transformation technique,[5],

$$W = V_{nt\times(nt-r)}\bar{W} \quad (66)$$

The eigenvalue equation of static buckling can be rewritten as,

$$[\bar{K} - N_0\bar{B}]\bar{W} = 0 \quad (67)$$

Where $\bar{W}$ is an eigenvector, and:

$$\bar{K} = V^T_{(nt-r)\times nt} K V_{nt\times(nt-r)} \quad (68)$$

Which V is an orthogonal matrix. by solving equation 63 using standard routines of eigenvalue solvers, static buckling values of composite laminates are computed.

Numerical example

A laminated composite plate of E-glass/epoxy materials with three layers is considered. Different static buckling loads with different ply angles for two cases of boundaries are computed. table(1). The in-plane compressive loads are applied in the x direction. The factor of buckling load is defined by $k=N_0b^2/\pi^2D_0$, which $D_0 = E_1h^3/[12(1-\nu_{12}\nu_{21})]$. The geometric parameters of material properties of the laminates are length a=b=1.0 m, thickness h=0.06 m, ratio of elastic constant $E_1/E_2=2.45$, ratio of elastic constant $G_{12}/E_2=0.48$, Poisson's ratio $\nu_{12}=0.23$. the buckling factors of laminates for simply supported and clamped boundaries are calculated using 15×15 nodes regularly distributed in the plate domain. The buckling values of laminates with clamped boundaries are generally larger than those with simply supported boundaries. For simply supported boundaries (SSSS), the buckling value increases as the ply angle increases but, for clamped boundaries (CCCC) the buckling value decreases as the ply angle increases. Nodal distribution in a laminate is shown in fig. (3).

Table 1- Buckling factor, k, for composite laminates with SSSS and CCCC boundaries, and different ply angles.

Ply angle	SSSS	Exact	CCCC	Exact
	K	K	K	K
(0°,0°,0°)	2.3951	2.3950	6.7633	6.7626
(30°,30°,30°)	2.5812	2.5793	6.3551	6.3548
(45°,45°,45°)	2.6474	2.6469	6.1537	6.1529
(0°,90°,0°)	2.3955	2.3951	6.7634	6.7628

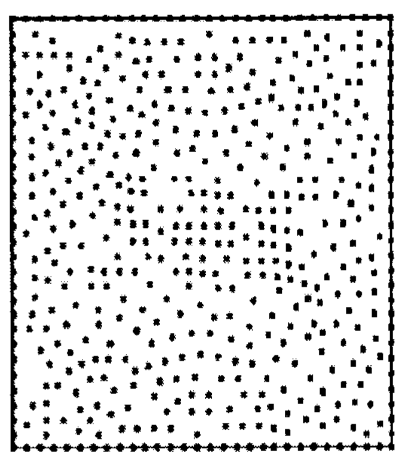

Figure 3- nodal distribution in a composite laminate

Conclusion

In this paper the element free Galerkin method which is a mesh-free method has been used to study composite laminates. Kirchoff plate theory is considered in analysis process. The dimension of the system matrix in EFG method is only one third of that in FEM method. There is no need for mesh generation in EFG method. only, a set of scattered nodes in the problem domain is required to construct the displacement shape functions. A numerical example is presented to show the efficiency of the EFG method.

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